Direct Methods to Systems of Linear Equations (Gauss elimination and Gauss – Jordan)

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Abstract

In this paper, a system of linear equations will be solved using matrices, and we will discuss the solution using the direct method, which is the method that leads to the exact solution after a limited number of simple arithmetic operations. The Gaussian elimination method will be used by converting the system into a trigonometric system and then using the back substitution method.

As for the second method, which is the Gauss-Jordan method for deletion, we perform a deletion process for a specific variable from all the equations after dividing an equation by the anchor element, which finally we get a matrix of diagonal equations, rather it is a matrix of unity.

Keywords:

Linear Equation , Matrix , Unit matrix , Diagonal matrix , Pivot element , Determinant

Many problems of numerical analysis are reduced to the problem of solving a linear system or set of equations. Among these problems, for example: solving ordinary or partial differential equations by finite-difference methods, solving eigenvalue problems in mathematical physics, fitting of data by the least-squares method, Polynomial approximation. The use of matrix conventions is very useful in solving problems of systems of linear equations.

Suppose that a set of linear equations to be solved will be written in the form: Ax=y

Where A : Transaction matrix

A =

$$\begin{pmatrix} a_{11} & a_{21} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix}$$

x = It is the vector to find ….. X =

$$\begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

y = The vector of the known constants is on the right-hand side

$$\begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{pmatrix}$$

That is, the set of equations to be solved is:

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots\; a_{1n}x_n = y_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots\; a_{2n}x_n = y_2$$

.

.

.

$$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots\; a_{nn}x_n = y_n$$

We will limit our study to the case where there is only one solution x of the set for each vector not on the right side. This means that we will be limited to the case where the number of group equations is equal to the number of unknowns (equal to n), that is, when the coefficients matrix A is square . The matrix A in this case must be invertible so that the set (or system) has exactly one solution for each y vector.

For matrix A to be invertible, the condition must be satisfied

det(A) <> 0

where det(A) is the determinant of the matrix A

We can also in this case express the solution of the set Ax = y in terms of determinants with what is called Cramer's rule; However, the use of determinants is not useful from a practical point of view in solving linear systems because the calculation of one determinant is generally equal - in terms of the degree of difficulty - with solving the linear system. Therefore, we will not use determinants to solve linear systems. However, we will mention a method for calculating the values of the determinants (to use these values in other fields), and this method is based on a direct method for solving linear systems.

In general, numerical methods for solving linear systems can be divided into two main types:

1 − Direct Methods

2– Iterative Methods

1– Direct Methods:

These are the methods that lead - in the absence of rounding errors and other errors - to the exact solution yet. A limited number of simple arithmetic operations. Practically speaking, since the computer works with finite word lengths, direct methods do not usually lead to exact solutions. In fact, errors resulting from round-off, instability, and loss of significance may lead to inaccurate, possibly wrong, and completely useless results. And a large part of the numerical analysis is related to the study of these errors and the investigation of the reason and how they arise and the means of overcoming them and the search for ways that lead to reducing their totality. The main method used for direct solutions is the Gauss elimination (G) method, or the modified method: the Gauss-Jordan method. We will review these two methods

(a) **Gauss** (**G**) **Elimination method**

This method can be summarized in the following two steps for solving a system of linear equations:

(1) Transforming the given system into a "triangular system" equivalent to the original system, and "triangular system" means that the matrix of coefficients becomes a triangular matrix.

(2) Solve the resulting trigonometric system by the "back substitution" method.

We prefer to explain Almighty or how to implement these two steps with a simple example, and then we mention after this the Gaussian elimination method algorithm to solve any general linear system.

Example 1

Using the Gaussian elimination method, find the solution to the following set of linear coefficients:

$20x_1 + 15x_2 + 12x_3 = 0$ (1)

$6x_1 + 4x_2 + 3x_3 = 0$ (2)

$6x_1 + 3x_2 + 2x_3 = 6$ (3)

The solution :

Step one: **convert the given system into a parabolic trigonometric system**

Start by removing the variable x from the second and third equations using the first equation, as follows:

$20x_1 + 15x_2 + 12x_3 = 0$ (1)\`

$-6/20$ X (1) + (2) $-1/2x_2 - 3/5x_3 = 0$ (2)\`

$-6/20\,X\,(1) + (3)$ $-3/2x_2 - 8/5x_3 = 6$ (3)\`

And that is by keeping the first equation as it is, and adding to the second equation the first equation multiplied by

$$-a_{21} / a_{11} = -6/20$$

We added to the third equation the first equation multiplied by

$$-a_{31} / a_{11} = -6/20$$

Now we remove the variable x_2 from the third equation (3) using the second equation (2), by adding to the third equation the second equation multiplied by:

$$-a'_{32} / a'_{22} = -(-3/2)/(-1/2) = -3$$

The first and second equations remain the same:

$20x_1 + 15x_2 + 12x_3 = 0$ (1)`

$-1/2x_2 - 3/5x_3 = 0$ (2)`

$-3\,X\,(2)' + (3)'$ $\quad -1/5x_3 = 6$ (3)`

To solve this resulting trigonometric system, we start from the back, that is, we start with the last equation that gives us the value of x_3

$$x_3 = 6/(1/5) = 30$$

Then the second gives us the value of x_2

$$x_3 = (0 + 3/5 \;\; X \;\; 30) / (-1/2) = -36$$

Finally, the first equation gives us the value of x_1:

$$x_1 = (0 - 15\,X\,(-36) - 12\,X\,30) / 20 = 9$$

That is, the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_n \end{pmatrix} = \begin{pmatrix} 9 \\ -36 \\ 30 \end{pmatrix}$$

Now we will re-solve the same set of equations given by the same method of elimination, but with some modifications in their order or their coefficients.

We start by eliminating x_1 from the second and third equations using the first equation:

$x_1 + 3/4x_2 + 3/5x_3 = 0$ (1)`

-6 X (1)` + (2) $-1/2x_2 - 3/5x_3 = 0$ (2)`

$-6 \ X\,(1)` + (3)$ $-3/2x_2 - 8/5x_3 = 6$ (3)`

Notice that we divided all the coefficients of the first equation (1) by 20 to make the factor of x_1 equal to unity, and we got the equivalent equation (1)`. Then we used this equivalent equation (1)` to delete x from each of the equations (3) and (2) by multiplying equation (1)` by an appropriate number (6-) and then adding it to the equation from which $x_1$ is required to be deleted, and equation (1)` remains as it is .

Now we remove the variable x_2 from the third equation using the second equation, but it is preferable that we change the order of the second and third equations - if necessary - so that we make a coefficient x_2 , which has the greatest absolute value, and since 1/2 < 3 /2 So we change the order of equations (3)` and (2)`.

$x_1 + (3/4)x_2 + (3/5)x_3 = 0$
$(-3/2)x_2 - (8/5)x_3 = 6$
$(-1/2)x_2 - (3/5)x_3 = 0$

We multiply the second equation by (-2/3) to make the factor of x_2 equal to the unit, then we use the resulting equation here to remove x_2 from the third equation, by multiplying the second resulting equation by ½ and adding it to the third, keeping the first equation and the second resulting equation as they are.

$x_1 + (3/4)x_2 + (3/5)x_3 = 0$
$x_2 + (16/15)x_3 = -4$
$(-1/15)x_3 = -2$

We multiply the last equation by (-15) to make the factor of x_3 equal to unity, and thus we bring the three equations to the following form:

$x_1 + (3/4)x_2 + (3/5)x_3 = 0$
$x_2 + (16/15)x_3 = -4$
$(x_3 = 30$

And this set of equations – which represents a trigonometric system – is equivalent to the given original set (that is, it has the same solution), because each of the three operations :

- Multiply an equation by a constant.
- Multiply an equation by a constant and add it to another equation.
- Substitution of two equations in place of each other.

(The operations that we performed on the original) set do not change the solution of the set .

The second step: **Solve the resulting trigonometric system by back–substitution method**.

We start solving the trigonometric system – that is, finding the values of the variables x_1 x_2 x_3 – using the last equation (ie the third) which gives the value of x_3 :

$x_3 = 30$

Then the penultimate equation (that is, the second), from which we get the value of x_2

$x_2 = -4 - 32 = -36$

Finally, the first equation maximizes the value of x_1

$x_1 = 27 - 18 = 9$

Thus, we obtained the same solution as before.

The process of making the coefficient x_i in Equation No. i equal to unity is called the normalization process. And the process of switching equations to make the x_i factor in equation No. i with the largest absolute value is called the partial pivoting process.

Now suppose we have a general system Ax=y

where A is the matrix of coefficients, x is the vector to find, and y is the vector of the known constants on the right side.

And we assume that |A| <> 0 , y <> 0

Gauss Elimination Algorithm

Step1: **converting the given system into a trigonometric system**

This can be done in one of two ways:

(**a**) **Elimination with Normalization**

- Divide the first equation by a_{11} . This element is called pivot element.

(If $a_{11} = 0$ it changed the order of the equations, that is, replace the first equation with another equation so that it becomes $a_{11} <> 0$).

- Add this first equation, multiplied by:

$$-a_{21}, -a_{31},,,,,,-a_{n1}$$

To Equation No.:

2,3,………n

respectively, in order to delete x_1 each of the equations from the second to the last.

- Divide the second equation by the new coefficient $a`_{22}$. This element in the main diameter, which we divide the equation by, is called a pivot element (also if it is equal to zero, change the order of the equations).
- Eliminate x_2 from all equations from the third to the last, in a similar way to the method of elimination x_1 , that is, by adding the new second equation multiplied by:

$$-a`_{32}, -a`_{42},,,,,,-a`_{n2}$$

To Equation No.:

3,4,………n

In order (note:

$$a`_{ij}$$

are the coefficients in the new equations).

- Continue the process of deleting the items $x_3, x_4, x_5, \ldots\ldots x_{n-1}$ in order, as much as possible.

(**b**) **Deletion without alteration**

- Add the first equation multiplied by

$$-a_{21} / a_{11}, -a_{31} / a_{11}, \ldots\ldots\ldots -a_{n1} / a_{11}$$

To Equation No.:

2,3, …… n

respectively, in order to delete x_1 from each of the equations from the second to the last.

- Eliminate x_2 from each of the equations from the third to the last in a similar way to the method of elimination x_1, that is, by adding the new second equation multiplied by

$$-a`_{32} / a`_{22} , -a`_{42} / a`_{22} , \ldots\ldots\ldots -a`_{n2} / a`_{22}$$

to equation No
3,4, …….. n

Respectively .

- Continue the process of removing the items $x_3, x_4, x_5, \ldots\ldots x_{n-1}$ in order.

Finally, we get a trigonometric system :

$$c_{11}x_1 + c_{12}x_2 + ,,,,,,,,,,,,, c_{1n}x_n = z_1$$
$$c_{22}x_2 + c_{23}x_3,,,,,,,,,,,,, c_{2n}x_n = z_2$$
$$\vdots$$
$$c_{n-1,n-1}x_{n-1} + c_{n-1,n}x_n = z_{n-1}$$
$$c_{n,n}x_n = z_n$$

note :

In case (a) (elimination with adjustment) the diagonal coefficients c_{ii} are equal to the unit.

That is:

$$c_{ii} : i = 1,2,3,\ldots\ldots\ldots n$$

Step 2: **Back-substitution to solve the resulting trigonometric system**

By substitution, we get the values of the variables in their inverse order, like this:

(from the last equation)

$x_n = z_n / c_{n,n}$

$x_{n-1} = (z_{n-1} - c_{n-1,n}x_n) / c_{n-1,n-1}$

…………

(from the penultimate equation)

$x_i = (z_i - \sum_{k=i+1}^{n} c_{i,k}x_k) / c_{ii}$

We can summarize the solution algorithm by back-substitution of the trigonometric system

Cx = z

Where C an upper trigonometric matrix (n x n) all of its diagonal elements are not equal to zero, with the following relationships:

$$x_i = (z_i - \sum_{k=i+1}^{n} c_{i,k} x_k) / c_{ii}$$

i = n , n-1 , …… 1

Noting that when i = n the sum $\sum_{k=i+1}^{n}$ becomes $\sum_{k=n+1}^{n}$, that is, there are no limits in this sum, and therefore it gives zero (conventionally).

Number of Operations in Gauss Elimination method.

If we use the Gaussian elimination method to solve a linear system consisting of n equation in n unknown, it can be shown that:

The number of multiplication and division operations

$$(n^3 + 3n^2 - n) / 3$$

The number of addition / subtraction operations:

$$(2n^3 + 3n^2 - 5n) / 6$$

That is, the number of multiplication and division operations (as well as addition and subtraction) is approximately equal

$$(n^3 / 3)$$

Gauss Elimination With Partial Pivoting

The anchor or wedge element in any step of this algorithm may not be equal to zero, and therefore can be divided by it, but it may be very small and thus may lead to large errors. A very small coefficient usually arises as a difference between two approximately equal numbers, and such errors can be avoided by changing the equations, i.e. by changing their arrangement (permutations) . It is preferable to make this arrangement so that the pivotal element - which was divided on it - becomes large. We usually rearrange the equations so that the pivotal element is the factor with the largest absolute value . Usually, the Gaussian method in which we follow this procedure is called the Gaussian method of deletion with partial pivoting.

The previously explained posterior compensation method leads to the following result:

theory:
If C it is a upper trigonometric matrix and all its diagonal elements are not equal to zero, then the matrix is invertible, that is, it has an inverse. C^{-1}
proof :
In fact, the aforementioned back-compensation method shows that the trigonometric system
Cx = z
has at most one solution for each given vector z.
Therefore,C it must be reversible based on the following known theory.

theory:
If we assume A that a matrix is square n x n , then all of the following statements are equivalent:
(a) A homogeneous system Ax = 0 has only a trivial solution x = 0
(b) For each vector y given on the right-hand side, the system Ax = y has a solution
(c) The matrix A is invertible.
Therefore, the vector x given by the previous back compensation algorithm
It is called: The solving the trigonometric system
Not: a solution to the trigonometric system

(B) Gauss–Gordon (G – J) Elimination Method

This method is the same as the Gauss delete method with a slight modification. For example, in the Gauss method of deletion with adjustment, after we divide an equation by the dependent element, we use this equation to delete a specific variable from all the equations below this equation. In the Gauss-Jordan method, we perform the process of deleting this variable expression from all the equations above In addition to that below this equation.

Therefore, the Gauss-Jordan method algorithm is the same as that of the Gauss method with this simple modification (elimination from all equations above and below the wedge element) and in this modified method we finally get a diagonal coefficient matrix, rather it is a Unit matrix and thus we directly get the solution The final set of equations without any other calculations, we don't need to apply the back compensation method.

Example 2

Using the Gauss-Jordan elimination method, find the solution to the set of linear equations given in Example 1

The solution :

We start the solution as in solving Example 1 where we delete x_1 from each of the second and third equations.. and when we reach the process of deduction x_2, it is not deleted from the third equation only, but from the first as well (using the second equation as well). Thus, instead of the following three equations (which we obtained while solving Example 1).

$$\begin{aligned} x_1 + (3/4)x_2 + (3/5)x_3 &= 0 \\ x_2 + (16/15)x_3 &= -4 \\ (-1/15)x_3 &= -2 \end{aligned}$$

We get the following three equations (where it was deleted x_2 from the first equation, by multiplying the second by (-3/4) and adding it to the first).

$$\begin{aligned} x_1 \qquad - (1/5)x_3 &= 3 \\ x_2 + (16/15)x_3 &= -4 \\ (-1/15)x_3 &= -2 \end{aligned}$$

Then we multiply the last equation by (–15) making my coefficient x_3 equal to the unit, and we use the resulting equation to delete x_3 from each of the first and second equations, by multiplying the third (resulted) by 1/5 and adding it to the first, and multiplying the third by –16/15 and adding it to the second, thus:

$$\begin{aligned} x_1 &= 9 \\ x_2 &= -36 \\ x_3 &= 30 \end{aligned}$$

These three equations give the required solution all in terms of matrices instead of equations as follows (and we can also follow this method of writing when solving using Gauss's elimination method):

x_1	x_2	x_3	Y
20	15	12	0
6	4	3	0
6	3	2	6

⟶

1	3/4	3/5	0
1	–1/2	–3/5	0
1	–3/2	–8/2	6

1	3/4	3/5	0
1	–3/2	–8/5	6
0	–1/2	–3/5	0

⟶ x–2/3

1	3/4	3/5	0
0	1	16/15	–4
0	–1/2	–3/5	0

x(1/2) , –(3/4)

1	0	–1/5	3
0	1	16/15	–4
0	0	–1/5	–2

⟶

x-(1/15)	1	0	-1/5	3
	0	1	16/15	-4
	0	0	1	30

→

X(1/5),-(16/15)	1	0	0	9
	0	1	0	-36
	0	0	1	30

→

x_1	9
x_2	-36
x_3	30

note :

If we suppose that the number of equations represents the same number of unknowns (that is, the variables), then the number of operations in the Gauss method - as we have seen previously - is estimated at about $n^3/3$, while the number of operations in the Gauss-Jordan method is about $n^3/2$. Therefore, the Gauss method is scientifically preferred for this reason, and the following can be proven:

The number of arithmetic operations in the Gauss-Jordan method:

The number of multiplication/division operations:

$$n^3/2 + n^2 - (n \ /2)$$

Number of addition / subtraction operations:

$$n^3/2 - (n \ /2)$$

That is, the number of multiplication and division operations (as well as addition and subtraction) is approximately equal.

$$n^3/2$$

Applications on Elimination Method

(a) Evaluating Determinants

From the properties of determinants, we know that no step of the Gaussian (or Gauss-Jordan) deletion algorithm changes a specific value of the coefficients matrix except for the process of dividing by each center element, and the process of switching two rows (or two columns). And since the value of the determinant of the resulting matrix is equal to 1 (note - in either of the two methods of deletion - that the resulting matrix has a zero inferior triangle, and ones in the main diagonal), then the value of the original determinant is thus equal to the product of pivots, with the sign of changing The result is if it has performed an odd number of row and column switch operations. .

det(A) = $(-)^k p_1 \ x \ p_2 \ x \ ....x \ p_n$

Where: k represents the number of times two rows or two columns are switched.

And $p_1 , p_2 , p_3 ...... p_n$ focal elements.

note:

Calculating the definite value of an nxn matrix by following the Gauss method of elimination requires the number of steps or operations to be estimated at $(n)^3 / 3$, while this number rises to the limits of !n when following the usual methods for calculating determinants.

Before we take an example of calculating determinants by the method of deletion, we study the second application of this method, which is the inversion of matrices, and then we give an example of the two applications together.

(**b**) **Finding Matrix Inversion**

We assume that A , Z , I , Square Matrices n x n and A.Z = I , and Z is the inverse of the matrix A, that is Z = $(A)^{-1}$, the matrix A is non-singular.

(We assume that A is the matrix whose inverse is to be found).

The relationship AZ = I can be written in detail as:

$$\begin{pmatrix} a_{11} & a_{21} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{pmatrix} \qquad \begin{pmatrix} z_{11} & z_{21} & . & . & . & z_{1n} \\ z_{21} & z_{22} & . & . & . & z_{2n} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ z_{n1} & z_{n2} & . & . & . & z_{nn} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & 1 \end{pmatrix}$$

From the properties of matrices, this relationship gives us the following set of relations:

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} z_{11} \\ z_{21} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} z_{12} \\ z_{22} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

....,

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} z_{1n} \\ z_{2n} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

Each of these relationships (numbered n) represents a set of linear equations (there are n equation).

That is, if we put Z = $(A)^{-1}$, then AZ = I

$A z_j = e_j \quad j = 1,2,3, \ldots\ldots n$

where Zj: column represents the number j in the matrix Z.

e_j : represents column number j in matrix I. That is:

$$z_j = \begin{pmatrix} z_{1j} \\ z_{2j} \\ . \\ . \\ . \\ . \\ . \\ . \\ z_{nj} \end{pmatrix}, \quad e_j = \begin{pmatrix} 0 \\ 0 \\ . \\ 1 \\ . \\ . \\ 0 \end{pmatrix}$$

Thus, the columns in the matrix $(A)^{-1} = Z$ are solutions to groups of linear equations, where the sides of the minus in these groups are equal to the columns of the unit matrix, and the coefficients matrix in each of these groups is equal to the matrix A (the inverse of which is required to be found)

. In the Gaussian deletion method (or Gauss – Jordan), we can deal with several Yemeni parties at the same time. If we apply the sit-elimination method on the system AZ = I, so that we treat the columns of the unit matrix 1 – at the same time – as several vectors No (that is, several vectors on the right sides) and the columns of the matrix Z as the corresponding solution vectors (x), We thus obtain – from these solution vectors – the inverse matrix to be found $(A)^{-1} = Z$, where these vectors are the columns of that matrix.

Example 3

Suppose that :

$$A= \begin{bmatrix} 2 & -7 & 4 \\ 1 & 9 & -6 \\ -3 & 8 & 5 \end{bmatrix} \quad Y = \begin{bmatrix} 9 \\ 1 \\ 6 \end{bmatrix}$$

(I) Find, using the Gauss-Jordan method, the solution of the linear system Ax = y (without switching any rows or columns, and using ordinary fractions in the calculations).

(II) Calculate the elements of the inverse matrix A^{-1} with the avarice of the system AZ = I, using partial angularity.

(III) Find the value of |A|

The solution:

(I), (II) Since the coefficient matrix A will be used to find both the solution of the linear system Ax = y , and the inverse matrix A^{-1}, so we solve the parts (1) and (i) of the question together.

X (1/2)	2	-7	4	9	1	0	0
	1	9	-6	1	0	1	0
	-3	8	5	6	0	0	1

	1	-7/2	2	9/2	1/2	0	0
X (2/25)	0	25/2	-8	-7/2	-1/2	1	0
	0	-5/2	11	39/2	3/2	0	1

	1	0	-6/25	88/25	9//25	7/25	0
X (5/47)	0	1	-16/25	-7/25	-1/25	2/25	0
	0	0	47/5	94/5	7/5	1/5	1

1	0	0	4	93/235	67/235	6/235
0	1	1	1	13/235	22/235	16/235
0	0	0	2	7/47	1/47	5/47

$$X = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \quad A^{-1} = 1/235 \begin{bmatrix} 93 & 67 & 6 \\ 13 & 22 & 16 \\ 35 & 5 & 25 \end{bmatrix}$$

(III) Since no two rows or any two columns have been replaced (k = 0), then: The value of the determinant = the product of the pivot elements.

Det(A) = |A| = 2 x 25/2 x 47/5 = 235

The number of arithmetic operations required to find the inverse of a matrix.

In the previous example, we followed the Gauss-Jordan method of deleting to find the inverse of the matrix A , by obtaining on the left-hand side at the end the unit matrix, resulting in the right-hand side being the inverse matrix A^{-1}. It was possible to follow the seating method for deletion instead of the Gauss-Jordan method by getting in The left side at the end on a superimposed trigonometric matrix U and then we apply the backcompensation algorithm n times. It can be proven that: .

(a) If we follow the Gauss-Jordan method to find the inverseA^{-1}:

The number of multiplication / division operations:

$$(3/2)n^3 - (1/2)(n)$$

The number of addition / subtraction operations:

$$(3/2)n^3 - 2(n)^2 + (1/2)n$$

(b) Whereas if we follow the Gaussian method of elimination to find the inverse A^{-1}: the number of multiplication / division operations:

$$(4/3)n^3 - (1/3)(n)$$

The number of addition / subtraction operations

$$(4/3)n^3 - (3/2)n^2 + (1/6)n$$

That is, in the Gauss-Jordan method for finding the inverse A^{-1}, the number of multiplication and division operations (as well as the number of addition and subtraction operations) is approximately equal $(3/2)n^3$, while the corresponding number in the Gaussian method of deletion is $(4/3)n^3$

(**C**) **Factorization of a Matrix**:

Sometimes you need to decompose a square matrix A into the product of two matrices: one is a lower triangular matrix (L) and the other is an upper triangular matrix (U).

That is:

A = LU

Sometimes this process of decomposition is called triangular decomposition. Sometimes there are conditions for the elements of the matrix L or the elements of the matrix U.

In order to know how to take advantage of the method of elimination in the work of this analysis, we suppose that we have n equations in n unknowns:

Ax = y

where A is an n x n matrix. After one step of the elimination process, matrix A1 becomes:

$$\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a`_{22} & . & . & . & a`_{2n} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ a_{n1} & a`_{n2} & . & . & . & a`_{nn} \end{pmatrix}$$

Assuming that there are no substitutions. As we know, we have obtained matrix A1 from matrix A, by subtracting the first row multiplied by (a_{i1}/a_{11}) from row No i And we can retrieve the matrix A from the matrix A1 by adding the first row multiplied by (a_{i1}/a_{11}) to row No. i where

i = 1, 2, ..., n . This is equivalent to the multiplication:

A = $L_1 A_1$

L_1 =

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 1_{21} & 1 & . & . & . & . \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 1_{n1} & 0 & . & . & 0 & 1 \end{pmatrix}$$

$I_{i1} = (a_{i1}/a_{11}) \quad , i = 2,3,\ldots..n$

Similarly, the new right-hand side y_1 related to its old value y by the relation:

y = $L_1 y_1$

Assuming that there are no permutations, the second step in the elimination process is to create the matrix A_2 =

$$\begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ 0 & a`_{22} & . & . & . & a`_{2n} \\ 0 & 0 & a``_{33} & & & a``_{2n} \\ . & & & & & \\ . & & & & & \\ 0 & 0 & a``_{n3} & . & . & a``_{nn} \end{pmatrix}$$

By subtracting the second row

$a`_{12} / a`_{22}$ from row 1 for the values i= 3,4,,,,,, n . Once again we can restore A_1 with the inverse process: $A_1 = L_2 A_2$

And so the

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & & & 0 \\ . & & & & & \\ . & & & & & \\ 0 & 1_{n2} & 0 & . & . & 1 \end{pmatrix}$$

$1_{i2} = (a`_{i2} / a`_{22})$, I = 2,3,4 ,,,,,,,n

And so the

A = $L_1 A_1 = L_1 L_2 A_2$

Likewise

y = $L_1 y_1 = L_1 L_2 y_2$

where y_2 is the right side after two steps.

Assuming that there are no substitutions, by repeating the above we can write:

A = $L_1 L_2 \ldots\ldots\ldots L_{n-1} U$

y = $L_1 L_2 = L_{n-1} L_2 z$

U: the last superscript.

Z: the last right-hand side.

L_j: the lower trigonometric matrix.

L_j =

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & 1_{j+1,j} & & 0 \\ . & & & . & & \\ . & & & & & \\ 0 & 0 & . & 1_{n,j} & . & 1 \end{pmatrix}$$

1_{ij} : A multiple of row j which is subtracted from row i during the process of reduction.

Thus:

$$\begin{pmatrix} 1 & 0 & . & . & . & 0 \\ 1_{21} & 1 & . & . & . & 0 \\ 1_{31} & 1_{32} & 1 & 1 & & 0 \\ . & & & . & & \\ . & & & & & \\ 1_{n1} & 1_{n2} & . & . & 1_{n,n-1} & 1 \end{pmatrix}$$

where L is a lower trigonometric matrix.

Thus, we find that if the equations are not singular and if we do not need any permutations, then :.

A = LU

Y = Lx

where L is a lower trigonometric matrix whose diagonal elements are all ones.

and U is a superscript trigonometric matrix.

Then we say that the matrix A has been solved (or worked) into two trigonometric factors U and L:

Example 4

From solving the set of linear equations in Example 1-3 by Gaussian elimination method without any permutations of the equations (ie without partial angularity), find the matrix of coefficients decomposition into two trigonometric factors L, U.

The solution :

We can summarize the first step of pregnancy by deleting: without partial focus as follows:

20	15	12	0
6	4	3	0
6	3	2	6

→

	20	15	12	0
(−6/20)	0	−1/2	−3/5	0
(−6/20)	0	−3/2	−8/5	6

	20	15	12	0
	0	−1/2	−3/5	0
(−3)	0	0	1/5	6

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 6/12 & 1 & 0 \\ 6/20 & 0 & 1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/2/-1/2 & 1 \end{bmatrix}$$

$$L = L_1 L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3/10 & 1 & 0 \\ 3/10 & 3 & 1 \end{bmatrix}$$

From the last set of equations we have reached, we find that

U =

$$\begin{bmatrix} 20 & 25 & 12 \\ 0 & -1/2 & -3/2 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Z =

$$\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

It is easy to verify that A = LU , where

A =

$$\begin{bmatrix} 20 & 25 & 12 \\ 6 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix}$$

So is the

Lz =

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/10 & 1 & 0 \\ 3/10 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = y$$

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